Enabling Source Channel Separation for Communication Networks : The Uplink Case

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***Abstract*— This paper focuses on obtaining an outer bound on the performance of a discrete memoryless multi-source, single destination network. This outer bound is found to be achievable using separate source and channel coding for certain example channels. However, separate coding is not always optimal for such networks.**

In particular, the paper focuses on the multiple access channel with two correlated sources. This paper uses the outer bound on the capacity region and shows that the outer bound can be achieved by separate source and channel coding for the binary erasure and binary symmetric multiple access channels and the physically degraded relay channel.

I. Introduction

A multi-source, single destination network is an integral part of many communication systems today. The message sources in such a network are often correlated. Such correlations between different transmitter messages may be induced either naturally or by other means. For example, in a sensor correlated bed where a group of sensors monitor some property like temperature of a region, the signals from various sensors are correlated. Satellite communication system is another example where there is correlation in the observations of different satellites. In other cases, noisy versions of the same signal may be observed at different places or instants. In such cases, the correlation between different observations is not induced by any natural phenomenon.

These networks are difficult to analyze, as in most situations, joint source-channel coding has to be performed in order to fully utilize the correlation between the transmitter messages. Performing joint source and channel coding is not an easy task and it can significantly increase the complexity of the system. As a result, a lot of work has been done in trying to separate source coding from channel coding. The separation of source coding from channel coding makes the design of channel encoder and decoder independent of the source encoder and decoder and hence greatly reduces the complexity of the system.

In 1948, Shannon established the source-channel separation principle in a point to point communication system [1]. Since then, source-channel separation has been established for other generalized channels and sources [2][3]. Verdú et.al. characterize the class of channels for which separation theorem holds irrespective of the source statistics [2]. Recently, Song and Yeung establish the separation theorem for single source network coding [3]. However, the source channel separation established by Song is valid only for single sources. Also, the uplink network (also called the MAC channel) is not a part of the framework considered by Verdú [2] and Song [3]. Source-channel separation has also been studied in a network setting under various specific models on network characteristics [4], [5], [6], [7].

This paper has two main contributions :

1) We derive an outer bound for the capacity of a multiple access, memoryless network with single destination and with correlated sources. Such a network is shown in Figure 1.
2) We show that this outer bound can be achieved using separate source and channel coding in some situations like the binary erasure and the binary symmetric MAC channel and the physically degraded relay channel. We also give an example (Gaussian MAC channel) where our separate source and channel coding scheme does not achieve the outer bound.

The rest of the paper is organized in the following manner. In Section II, we describe the system model and the notations that will be used in the paper. Section III presents the main results of the paper. Section IV contains some concluding remarks. Finally, the proofs are given in the Appendix.

II. System Model

We adopt the following notation throughout the paper. S denotes any subset of $\{1, \ldots, M\}$, while S^c is its complement. $X(S)$ or X_S denotes the set of random variables $\{X(i) : i \in S\}$.

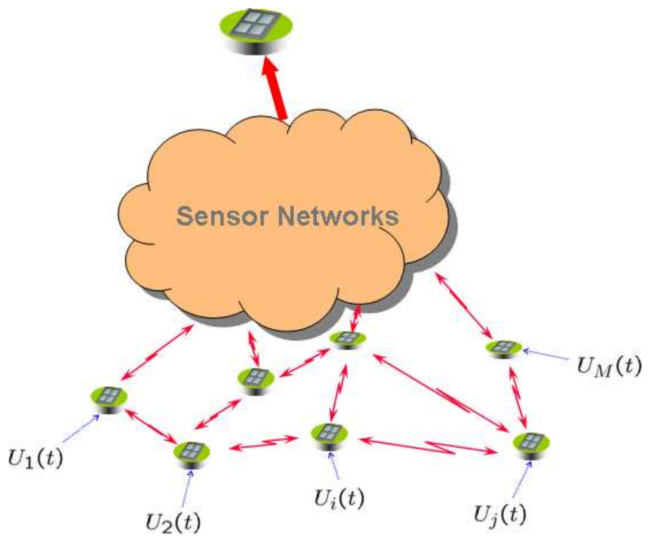

Fig. 1. Multi-Access Discrete Memoryless Network

Random variables (RVs) will be denoted by capital letters, while their realizations will be denoted by the respective lower case letters. The alphabet of a random variable X will be designated by a calligraphic letter $\mathcal{X}$, and that of the n-fold Cartesian power of $\mathcal{X}$ will be denoted as $\mathcal{X}^n$.

We consider a multiterminal network which consists of $M+1$ nodes. Let $\mathcal{V}$ denote the set of all the nodes, $\mathcal{V} = \{v_0, v_1, \ldots, v_M\}$. Node v_0 is the only sink in the network and intends to reproduce all the information sources in the network. Node v_i $(i = 1, \ldots, M)$ has an associated transmitted variable X_i and a received variable Y_i and node v_0 only has a received variable Y_0. The channel then is represented by the channel transition function $p(y_0, y_1, \ldots, y_M|x_1, \ldots, x_M)$, which is the conditional probability mass function of the outputs given the inputs. We further assume the channel is memoryless, namely,

$$p(y_0^N, y_1^N, \ldots, y_M^N|x_1^N, \ldots, x_M^N) = \prod_{n=1}^{N} p(y_{0,i}, y_{1,i}, \ldots, y_{M,i}|x_{1,i}, \ldots, x_{M,i}) \quad (1)$$

At node v_i $(i = 1, \ldots, M)$, a random variable $U_i \in \mathcal{U}_i$ is observed $(i = 1, \ldots, M)$, drawn i.i.d. from a known joint distribution $p(U_1 U_2 \ldots U_M)$ with finite alphabet. For the nodes that have no sources to transmit, $\mathcal{U}_i$ is empty, or in other words, $|\mathcal{U}_i| = 0$. Figure 2 describes the channel model.

In this paper, we obtain an outer bound for the capacity region of the system described above. We also show that this outer bound is achievable for a binary erasure MAC channel, a binary symmetric MAC channel and a physically degraded relay channel by performing source and channel coding separately. In a binary symmetric MAC channel, the sources U and V are binary, and so

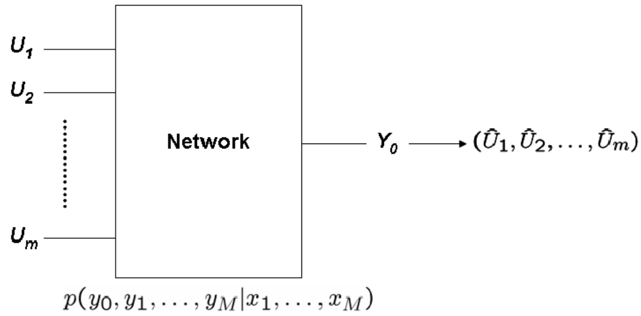

Fig. 2. The Channel Model of the Multi-Access Network

are the channel inputs and outputs. The channel output at the k^{th} time instant is given by $y_k = x_{1,k} \oplus x_{2,k}$ where '$\oplus$' represents the XOR operation. A binary erasure MAC channel is similar to the binary symmetric MAC channel, except that the channel output comes from the ternary alphabet $\mathcal{Y} = \{0, 1, e\}$, where the symbol e denotes an erasure. The channel output at time instant k is given by

$$Y_k = \begin{cases} X_{1,k} \oplus X_{2,k} & \text{with probability } 1-\epsilon \\ e & \text{with probability } \epsilon \end{cases} \quad (2)$$

A physically degraded relay channel [8] is described in figure (3). We have a source $\mathcal{W}$ that communicates with a destination. The discrete memoryless relay channel is denoted by $(\mathcal{X}_1 \times \mathcal{X}_2, p(y, y_1|x_1, x_2), \mathcal{Y} \times \mathcal{Y}_1)$. x_1 is the input of the channel while y is the output, y_1 is the relay's output and x_2 is the symbol chosen by the relay to transmit to the destination. In a degraded relay channel, $p(y, y_1|x_1, x_2) = p(y_1|x_1, x_2)p(y|x_2, y_1)$.

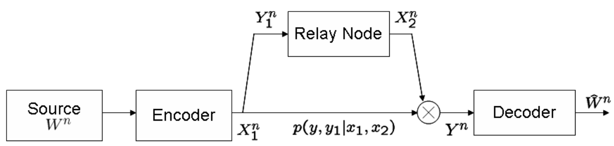

Fig. 3. Relay Channel

We also present an example (Gaussian MAC channel) where the outer bound is not achievable by our technique of separate source and channel coding. In a Gaussian MAC channel, the channel output at time instant k is described by

$$Y_k = X_{1,k} + X_{2,k} + Z_k \quad (3)$$

where, Z^n denotes a sequence of independent zero mean Gaussian random variable with variance σ^2. Also, in a Gaussian MAC channel, the transmitted codewords must satisfy the following power constraints.

$$\frac{1}{n}\sum_{i=1}^{n} X_j^2(i) \le P_j, \qquad j = 1, 2 \quad (4)$$

III. MAIN RESULTS

In this section, we present the main results that have been obtained in this paper. We give an outer bound on the capacity of a multi-source, single destination network. The proofs will be given in the appendix.

Theorem 3.1: An outer bound on the capacity region of the multi-source, single destination network is given by

$$H(U_S|U_{S^c}) \leq I(X_S; Y_{S^c}|X_{S^c}) + \epsilon_n \tag{5}$$

for any joint probability distribution $p(X_1, X_2, \ldots, X_m)$ and for any subset $S \subset \mathcal{V} - \{v_0\}$.

This is similar to the cut-set outer bound discussed in [9]. The proof follows from Fano's inequality and is given in the Appendix. The following results show that source-channel separation holds for the binary erasure multiple access channel, the binary symmetric multiple access channel and the physically degraded relay channel.

Corollary 3.2: Source-channel separation holds for the binary erasure channel described in Section II. That is, the outer bound described in Theorem 3.1 can be achieved by separate source and channel coding.

Corollary 3.3: Source-channel separation holds for the binary symmetric MAC channel described in Section II.

Corollary 3.4: Source-channel separation holds for the physically degraded relay channel described in Section II.

The proofs of Corollaries 3.2, 3.3 and 3.4 are given in the Appendix. We prove that source coding (performed by random binning) and channel coding can be separated.

However, source-channel separation is not optimal for any multi-source, single destination network. An example where it is not optimal is the case of a Gaussian MAC channel (Figure 4) for two correlated sources in the presence of feedback.

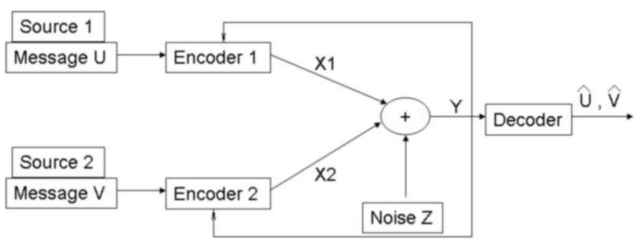

Fig. 4. Gaussian MAC channel for Two Correlated Sources

It can be shown that for a Gaussian MAC channel, the outer bound is given by the closure of

$$\begin{aligned} H(U|V) &\leq \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma^2}(1-\rho^2)\right) \\ H(V|U) &\leq \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma^2}(1-\rho^2)\right) \\ H(U,V) &\leq \frac{1}{2}\log\left(1 + \frac{P_1 + P_2 + 2\sqrt{P_1P_2}\rho}{\sigma^2}\right) \end{aligned} \tag{6}$$

for $0 \leq \rho \leq 1$. This outer bound is obtained from Theorem 3.1 and using Ozarow's outer bound [10]. It can be shown that for the case when $U = V$, that is when the sources are 100% correlated, only the last inequality remains. When we perform source coding by random binning, we get only one bin index, reducing the problem to a single transmit, single receive problem. It is thus impossible to achieve $\max_{p(X_1,X_2)} I(X_1, X_2; Y)$, which is in fact the capacity of this channel.

IV. CONCLUSIONS

In this paper, we obtain an outer bound for the multi-source, single destination network with correlated sources. We also established that this outer bound can be achieved by separate source and channel coding for the binary erasure MAC channel as defined in Section II, the binary symmetric MAC channel and the physically degraded relay channel. However, there are systems where our separate coding scheme is not optimal, that is, does not achieve the outer bound. The Gaussian MAC channel is an example of such a system. Further analysis is under progress to determine the optimal coding scheme for a Gaussian MAC channel.

V. APPENDIX

Here, we give the proofs of Theorem 3.1 and an outline for the proofs of Corollaries 3.2, 3.3 and 3.4.

A. Proof of Theorem 3.1

We need the following lemma for the proof.

Lemma 5.1 (Fano's inequality): If $P_e^{(n)} \leq 0.5$ and $S \subset \mathcal{V} - \{v_0\}$ is any subset,

$$H(U_S^n|U_{S^c}^n, Y_{S^c}^n) \leq H(P_e^{(n)}) + n|S|P_e^{(n)}\log(|\mathcal{U}| - 1)$$

Proof: Now for the cut on the set of nodes S, S^c, let the encoders in S cooperate and the decoders in S^c cooperate. Define the decoding error by using cooperative encoding/decoding as $\tilde{P}_{e,S}^{(n)}$,

$$\tilde{P}_{e,S}^{(n)} = Pr(U_S^n \neq \hat{\hat{U}}_S^n), \tag{7}$$

where $\hat{\hat{U}}_S^n$ is the estimate of U_S^n from Y_S^n. Obviously we have

$$\tilde{P}_{e,S}^{(n)} \leq P_e^{(n)} . \tag{8}$$

Next we define an error random variable,

$$E = \begin{cases} 1 , & \hat{\hat{U}}_S^n \neq U_S^n \\ 0 , & \hat{\hat{U}}_S^n = U_S^n \end{cases} , \tag{9}$$

Then we have

$$\begin{aligned} H(U_S^n|U_{S^c}^n, Y_{S^c}^n) &= H(E, U_S^n|U_{S^c}^n, Y_{S^c}^n) - H(E|U_{\mathcal{V}}^n, Y_{S^c}^n) && \text{(10a)} \\ &= H(E, U_S^n|U_{S^c}^n, Y_{S^c}^n) && \text{(10b)} \\ &= H(E|U_{S^c}^n, Y_{S^c}^n) + H(U_S^n|E, U_{S^c}^n, Y_{S^c}^n) && \text{(10c)} \\ &\leq H(P_{e,S}^{(n)}) + n|S|P_{e,S}^{(n)} \log(|\mathcal{U}| - 1) && \text{(10d)} \\ &\leq H(P_e^{(n)}) + n|S|P_e^{(n)} \log(|\mathcal{U}| - 1) , && \text{(10e)} \end{aligned}$$

where

1) (10a) and (10c) are the two different ways to expand $H(E, U_S^n|U_{S^c}^n, Y_{S^c}^n)$ using the chain rule for entropies;
2) (10b) is from the fact that E is a function of $U_{\mathcal{V}}^n, Y_{S^c}^n$;
3) (10d) follows from that the conditional decreases the entropy;
4) (10e) is due to (8) and that $H(P_e)$ is an increasing function when $P_e \leq 0.5$.

■

The outer bound can now be proved using the above lemma. For any subset $S \subset \mathcal{V} - \{v_0\}$, we have

$$\begin{aligned} H(U_S^n|U_{S^c}^n) &= I(U_S^n; Y_{S^c}^n|U_{S^c}^n) + H(U_S^n|U_{S^c}^n, Y_{S^c}^n) \\ &\overset{(a)}{\leq} I(U_S^n; Y_{S^c}^n|U_{S^c}^n) + n\epsilon_n && \text{(11)} \\ &= H(Y_{S^c}^n|U_{T^c}^n) - H(Y_{S^c}^n|U_{\mathcal{V}}^n) + n\epsilon_n \\ &\overset{(b)}{\leq} \sum_{k=1}^n H(Y_{S^c,k}|U_{S^c}^n, Y_{S^c}^{k-1}) - H(Y_{S^c}^n|X_{\mathcal{V}}^n) + n\epsilon_n \\ &\overset{(c)}{\leq} \sum_{k=1}^n \Big(H(Y_{S^c,k}|X_{S^c,k}) - H(Y_{S^c,k}|X_{S,k}, X_{S^c,k}) \Big) \\ &\qquad + n\epsilon_n \\ &= \sum_{k=1}^n I(X_{S,k}; Y_{S^c,k}|X_{S^c,k}) + n\epsilon_n , \end{aligned}$$

where

(a) follows from Fano's inequality proved in Lemma 5.1;
(b) follows from the fact that $U_{\mathcal{V}}^n \Rightarrow X_{\mathcal{V}}^n \Rightarrow Y_{S^c}^n$ forms a Markov chain;
(c) follows from the fact that $U_{S_c}^n \Rightarrow X_{S^c,k} \Rightarrow Y_{S^c,k}$ forms a Markov chain and dropping the conditionals increases the entropy for nonfeedback encoding, and follows from that fact that X_{S^c} is a deterministic function of $U_{S_c}^n$ and $Y_{S^c}^{k-1}$ for feedback encoding.

Therefore, dividing throughout by n, we get

$$H(U_S|U_{S^c}) \leq I(X_S; Y_{S^c}|X_{S^c}) + \epsilon_n \tag{12}$$

Therefore, if $H(U_S|U_{S^c}) \leq I(X_S; Y_{S^c}|X_{S^c})$ is not satisfied, then the asymptotic probability of error will be positive and hence reliable communication will not be possible.

B. *Proof of Corollary 3.2*

To prove this, we need the following lemma.

Lemma 5.2: For a binary erasure MAC channel described in Section II,

$$\begin{aligned} \max_{p(X_1,X_2)} I(X_1; Y|X_2) &= 1 - \epsilon \\ \max_{p(X_1,X_2)} I(X_2; Y|X_1) &= 1 - \epsilon && \text{(13)} \\ \max_{p(X_1,X_2)} I(X_1, X_2; Y) &= 1 - \epsilon. \end{aligned}$$

This is just a maximization problem and the proof is fairly simple. It is clear that $H(Y|X_1, X_2) = H(\epsilon)$. We just optimize $H(Y), H(Y|X_1), H(Y|X_2)$ over all joint distributions $p(X_1, X_2)$ to get the result.

Hence, the outer bounds given in Theorem 3.1 reduces to

$$\begin{aligned} H(U|V) &\leq 1 - \epsilon \\ H(V|U) &\leq 1 - \epsilon && \text{(14)} \\ H(U, V) &\leq 1 - \epsilon \end{aligned}$$

Clearly, only the last inequality is effective and the outer bound reduces to $H(U, V) \leq 1 - \epsilon$.

Achievability : Here, we present a separate source and channel coding scheme that achieves this outer bound. That is, under this coding scheme, the sources U and V can be reliably transmitted if the expressions in equation (14) are satisfied. The encoding and decoding are done in two stages - source coding and channel coding. The source coding is done by Slepian-Wolf binning.

Encoding : Let the sources U and V have entropies $H(U)$ and $H(V)$ respectively. We consider n length typical sequences from both the sources. Source U has approximately $2^{nH(U)}$ n- length typical sequences, and source V has approximately $2^{nH(V)}$ n- length typical sequences. Each n length typical sequence from source U is mapped to one of 2^{nR_1} bins in an arbitrary manner. That is, we take each typical sequence and map it to an arbitrary bin. Similarly, each typical sequence from source V is mapped to one of 2^{nR_2} bins in an arbitrary manner. This mapping or code-book is made available to both the source encoder and the source decoder. This encoding scheme is basically Slepian-Wolf binning [11].

The symbols from each source are taken in blocks of size n. Each block of symbols u^n and v^n are mapped into bin indices θ_1 and θ_2 respectively according to the code-book ($\theta_1 \in \{0, 1, \ldots, 2^{nR_1}\}, \theta_2 \in \{0, 1, \ldots, 2^{nR_2}\}$). It is clear that although u^n and v^n are correlated, the bin indices θ_1 and θ_2 are uncorrelated and independent. Each bin index θ_1 is mapped into $X_1^n(\theta_1)$ generated randomly by $\Pi_{i=1}^n p(X_{1,i})$. Similarly, every bin index θ_2 is mapped into $X_2^n(\theta_2)$ generated randomly by $\Pi_{i=1}^n p(X_{2,i})$. Here X_1^n and X_2^n are sequences of binary random variables.

We can see that, by silencing the second transmitter (making it transmit only zeros), we can achieve $R_1 \leq 1-\epsilon$ with arbitrarily low probability of error for sufficiently large block lengths. Similarly, by silencing the first transmitter, we can achieve $R_2 \leq 1 - \epsilon$ with arbitrarily low probability of error. And by time sharing, we can achieve $R_1 + R_2 \leq 1 - \epsilon$. Hence, the bin indices θ_1 and θ_2 are recovered accurately if the following condition is satisfied.

$$R_1 + R_2 \leq 1 - \epsilon \tag{15}$$

From Slepian Wolf decoding scheme [11][12], it is clear that once the bin indices are recovered accurately, the source estimates $\hat{u}^n$ and $\hat{v}^n$ are recovered accurately if

$$\begin{aligned} H(U|V) &\leq R_1 \\ H(V|U) &\leq R_2 \\ H(U,V) &\leq R_1 + R_2 \end{aligned} \tag{16}$$

Again, we can see that only the last inequality is effective. That is, the sources can be sent accurately if $H(U, V) \leq 1 - \epsilon$.

Hence, we have showed that the outer bound given in Theorem 3.1 is achievable by separate source and channel coding. Therefore, source-channel separation holds for the binary erasure MAC channel.

C. Proof of Corollary 3.3

The proof is similar to that of Corollary 3.2. We start of with the following lemma.

Lemma 5.3: For a binary symmetric MAC channel described in section II

$$\begin{aligned} \max_{p(X_1,X_2)} I(X_1; Y|X_2) &= 1 \\ \max_{p(X_1,X_2)} I(X_2; Y|X_1) &= 1 \\ \max_{p(X_1,X_2)} I(X_1, X_2; Y) &= 1 \end{aligned} \tag{17}$$

The proof is similar to that of Lemma 5.2 and is skipped.

The outer bound of Theorem 3.1 reduces to $H(U, V) \leq 1$. The achievability of the outer bound by separate source and channel coding is very similar to that of Corollary 3.2 and is omitted here.

D. Proof of Corollary 3.4

Here, we prove that source-channel separation is optimal for physically degraded relay channel. The capacity of the degraded relay channel [8] is

$$C = \sup_{p(x_1,x_2)} \min(I(X_1, X_2; Y), I(X_1; Y_1|X_2)) \tag{18}$$

where the supremum is over all the joint probability functions $p(x_1, x_2)$.

The outer bound of the relay channel is given by Theorem 3.1

$$H(W) \leq \min(I(X_1, X_2; Y), I(X_1; Y, Y_1|X_2)) \tag{19}$$

For a degraded relay channel, Y is independent of X_1 when (Y_1, X_2) are given, or $I(X_1; Y|Y_1, X_2) = 0$. Therefore, we have,

$$\begin{aligned} I(X_1; Y, Y_1|X_2) &= I(X_1; Y_1|X_2) + I(X_1; Y|Y_1, X_2) \\ &= I(X_1; Y_1|X_2) \end{aligned} \tag{20}$$

Hence, from (18), (19 and (20), we can see that the capacity is in fact the outer bound.

The encoding scheme is straightforward. Let W^n be a n - length sequence of message symbols from source $\mathcal{W}$ derived from a probability distribution $p(w^n)$. Each of $2^{nH(W)}$ typical sequences is mapped into one of 2^{nR} bins in a random manner. We denote the bin index by θ. Then, R can be thought of as the rate of the bin index. The bin index is coded into X^n and is transmitted. Therefore, the bin index can be recovered correctly if $R \leq \min(I(X_1, X_2; Y), I(X_1; Y_1|X_2))$ for any joint probability distribution $p(X_1, X_2)$. Then, the message estimate $\hat{W}^n$ can be recovered correctly if $R \geq H(W)$. Hence, the capacity region is given by $H(W) \leq \min(I(X_1, X_2; Y), I(X_1; Y_1|X_2))$. Hence, source-channel separation holds for the physically degraded relay channel.

REFERENCES

[1] C. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379–423, July 1948.

[2] S. Vembu, S. Verdù, and Y. Steinberg, "The source-channel separation theorem revisited," *IEEE Trans. Inform. Theory*, vol. 41, no. 1, pp. 44–54, Jan. 1995.

[3] L. Song, W. Yeung, and N. Cai, "A separation theorem for single source network coding," *IEEE Trans. Inform. Theory*, vol. 52, no. 5, pp. 1861–1871, May 2006.

[4] J. Barros and S. Servetto, "Network information flow with correlated sources," *IEEE Trans. Inform. Theory*, vol. 52, no. 1, pp. 155–170, Jan. 2006.

[5] ——, "On the capacity of the reachback channel in wireless sensor networks," *In Proc. IEEE Intl. Workshop Multimedia Sig. Proc., US Virgin Islands*, 2002.

[6] M. Effros, M. Médard, T. Ho, S. Ray, D. Karger, R. Koetter, and B. Hassibi, "Linear network codes: A unified framework for source, channel, and network coding," in *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, ser. Advances in Network Information Theory, G. et al., Ed., vol. 66.

[7] S. Ray, M. Effros, M. Médard, R. Koetter, T. Ho, D. Karger, and J. Abounadi, "On separation, randomness and linearity for network codes over finite fields," MIT, LIDS Technical Report 2687, Mar. 2006.

[8] T. M. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inform. Theory*, vol. 25, pp. 572–584, 1979.

[9] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. Wiley, 1991.

[10] L. H. Ozarow, "The capacity of the white Gaussian multiple access channel with feedback," *IEEE Trans. Inform. Theory*, vol. 30, no. 4, July 1984.

[11] D. Slepian and J. Wolf, "Noiseless coding of correlated sources," *IEEE Trans. Inform. Theory*, vol. 19, pp. 471–480, 1973.

[12] ——, "A coding theorem for multiple access channels with correlated sources," *Bell Labs Tech. J.*, vol. 52, pp. 1037–1076, 1973.